On the Controllability of an Air Hockey Puck *

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Abstract

In this paper we investigate the controllability of an air hockey puck subject to impulsive inputs, which is a special case of the more general problem of impulsive manipulation of rigid objects. We first formulate the control problem on the tangent bundle of the three dimensional configuration manifold of the system and characterize the set of velocities reachable with arbitrary impulsive inputs. In reality, one cannot specify impulsive forces as inputs but rather only the velocities of a mallet (or striker) impacting the puck. Therefore, we refine the characterization to include the case that the impulsive forces act on the puck through an impact with a second object. This characterization then depends on the material properties of the impacting objects. We use the two-dimensional Routh impact model which allows us to characterize the reachable velocities in terms of the coefficients of friction and restitution of the objects.

1 Introduction

The problem of *Impulsive Manipulation* is the problem of controlling trajectories of physical objects through impacts and arises in a number of robotic applications including juggling [8], hopping [5], and batting [3]. Impulsive manipulation also arises in the study of locomotion. For example, the analysis of human subjects indicates that visual regulation of step length in running over irregular terrain is achieved primarily by adjusting the vertical impulse imparted to the ground at each step [12].

Our own interest in this problem arises both from our interest in bipedal locomotion [10], and from our development of an air hockey playing robot [1, 2, 6]. In air hockey, a puck sliding on an air table is controlled by impacts from a mallet or striker. The control problem considered here differs from other problems involving impulsive manipulation of rigid objects [4] in that we allow only a single impact, i.e., the impulsive control can act only once. Under these conditions, arbitrary velocities cannot be achieved. For example changing the rotational velocity independently of the translational velocity requires a moment couple [4], which can be achieved by two impacts acting at distinct points on the puck but not by a single impact. We therefore seek to characterize precisely the reachable subset of velocities achievable with a single impact between the puck and mallet.

We first characterize the reachable set of velocities assuming an arbitrary impulsive input. Since these impulses arise from the impact between the puck and a striker, we next treat the

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velocity of the striker as the control input and refine the characterization of reachable puck velocities. As we shall see, this characterizatin depends on the material properties of the impacting objects. We use the Routh two-dimensional impact model [9] which models the material deformation in the direction normal to the impact and relative sliding between the impacting objects in tangential direction.

2 Modeling

Consider the setup shown in Figure 1 showing a circular puck with tangential and normal impulse components P_b and P_n, respectively, acting on the puck at the origin of an attached coordinate frame (b, n), with the frame (x, y) representing the world or base frame. We assume that the

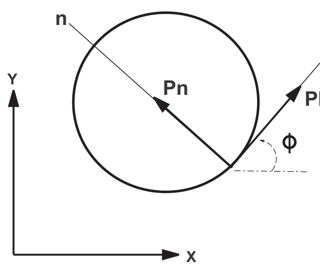

Figure 1: Coordinate Frames

puck is a uniform thin disk of mass m, radius r and moment of inertia, $I = \frac{mr^2}{2}$, about its central axis. The configuration of the puck at time t is given as $(x(t), y(t), \theta(t))^T$, where $x(t), y(t)$ are the coordinates in the base frame of the center of mass of the object, and $\theta(t)$ is the orientation of the object measured relative to the x-axis of the base frame. The *Configuration Space*, of the object is thus $\mathcal{SE}(2) = \Re^2 \times S^1$, where S^1 is the unit circle. Assuming that the air table surface is frictionless, level and planar, so that the effects of gravity can be ignored, a moving puck will travel along a straight line at constant velocity until it's trajectory is changed through impacts. The impulse vector $P = (P_b, P_n)^T$, is defined to be finite and results in a discontinuous change in linear and angular velocity of the objects which are otherwise constant between impacts.

For this reason we consider the control problem on the Velocity Space represented by $T\mathcal{SE}(2)$, the tangent bundle of $\mathcal{SE}(2)$. The motion of the puck in the (b, n)-frame is thus given by the impulse-momentum equations

$$m(\dot{b} - \dot{b}_0) = P_b \quad (1)$$

$$m(\dot{n} - \dot{n}_0) = P_n \quad (2)$$

$$\frac{mr^2}{2}(\dot{\theta} - \dot{\theta}_0) = rP_b \quad (3)$$

where $\dot{b}_0$, $\dot{n}_0$ $\dot{\theta}_0$ and $\dot{b}$, $\dot{n}$ $\dot{\theta}$ represent the velocities just before and just after the impact, respectively. The corresponding equations of motion in the base frame are

$$m(\dot{x} - \dot{x}_0) = \cos(\phi)P_b - \sin(\phi)P_n \quad (4)$$

$$m(\dot{y} - \dot{y}_0) = \sin(\phi)P_b + \cos(\phi)P_n \quad (5)$$

$$\frac{mr^2}{2}(\dot{\theta} - \dot{\theta}_0) = rP_b \quad (6)$$

3 Impulse Controllability

In this section we assume that the control inputs to the puck is the impulse vector (P_b, P_n) and the angle ϕ and we characterize the change in velocity that is achievable with a single impulse and choice of angle ϕ. Set $\Delta\dot{x} = \dot{x} - \dot{x}_0$, $\Delta\dot{y} = \dot{y} - \dot{y}_0$, $\Delta\dot{\theta} = \dot{\theta} - \dot{\theta}_0$. Then the equations of motion in terms of velocity differential are

$$\Delta\dot{x} = \frac{\cos\phi}{m}P_b - \frac{\sin\phi}{m}P_n \quad (7)$$

$$\Delta\dot{y} = \frac{\sin\phi}{m}P_b + \frac{\cos\phi}{m}P_n \quad (8)$$

$$\Delta\dot{\theta} = \frac{2}{mr}P_b \quad (9)$$

which we write as

$$\Delta V = B(\phi)U \quad (10)$$

where $\Delta V = (\Delta\dot{x}, \Delta\dot{y}, \Delta\dot{\theta})^T$, $U = (P_b, P_n)^T$ and

$$B(\phi) = \begin{bmatrix} \frac{\cos(\phi)}{m} & -\frac{\sin(\phi)}{m} \\ \frac{\sin(\phi)}{m} & \frac{\cos(\phi)}{m} \\ \frac{2}{mr} & 0 \end{bmatrix} \qquad (11)$$

We leave it as a simple exercise to show that the range of the mapping $B(\phi)$, for a fixed ϕ is given by

$$\begin{aligned} \text{Range}(B(\phi)) = \{(V_1, V_2, V_3)^T \in \Re^3 | \\ V_1 \cos\phi + V_2 \sin\phi - \tfrac{r}{2} V_3 = 0\} \end{aligned} \qquad (12)$$

We can now state the following result which characterizes the set of reachable velocities. By reachability we mean reachability from the origin for the system (10).

Theorem 1: The set, $\mathcal{R}$, of reachable puck velocities is the half space

$$\mathcal{R} = \{(V_1, V_2, V_3)^T \in \Re^3 \mid V_1^2 + V_2^2 - \frac{r^2}{4} V_3^2 \geq 0\} \qquad (13)$$

In other words, given a desired velocity vector $V = (V_1, V_2, V_3)^T$ satisfying (13) there exist inputs $U = (P_b, P_n)^T$, ϕ satisfying (12) and

$$B(\phi)U = V \qquad (14)$$

Proof: First, if both V_1 and V_2 are zero, then from both (12) and (13) we must have V_3 is zero. In this case set $P_b = P_n = 0$ and let ϕ be arbitrary. If not both V_1 and V_2 are zero, choose α such that

$$\cos(\alpha) = \frac{V_1}{\sqrt{V_1^2 + V_2^2}} \; ; \; \sin(\alpha) = \frac{V_2}{\sqrt{V_1^2 + V_2^2}} \qquad (15)$$

Dividing (12) through by $\sqrt{V_1^2 + V_2^2}$ and using a trig identity, we can write the constraint equation (12) as

$$\cos(\phi - \alpha) = C \qquad (16)$$

where $C = \frac{\frac{r}{2} V_3}{\sqrt{V_1^2 + V_2^2}}$. Equivalently we may express this as

$$\tan(\phi - \alpha) = \frac{\sqrt{V_1^2 + V_2^2 - \frac{r^2}{4} V_3^2}}{\frac{r}{2} V_3} \qquad (17)$$

Then, if and only if (13) is satisfied, the angle ϕ satisfying (12) is given by

$$\phi = \tan^{-1}(\frac{V_2}{V_1}) + \tan^{-1}(\frac{\sqrt{V_1^2 + V_2^2 - \frac{r^2}{4} V_3^2}}{\frac{r}{2} V_3}) \qquad (18)$$

Now, choose

$$P_b = \frac{mr}{2} V_3 \qquad (19)$$
$$P_n = m(V_2 \cos\phi - V_1 \sin\phi) \qquad (20)$$

Then a direct calculation shows that substituting (19)-(20) into (7)-(9) yields (14) provided the constraint (12) is satisfied.

4 Impact Controllability

In this section we discuss the case where the impulsive forces to the puck are produced by impacts with a striker. Effectively we take as control inputs the velocity of the striker. We utilize the Routh two-dimensional impact model to determine the impulses in terms of the object velocities [9]. Consider the impact event shown in

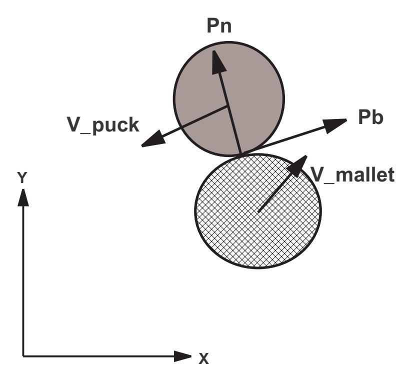

Figure 2: puck/wall impact

Figure 2. We make the simplifying assumption that the inertia of the striker is much larger than that of the puck so that the striker velocities are not changed by the impact. We also assume that

the rotational velocity of the striker is zero[1].

The Routh method [9] separates the normal impulse, P_n, into two parts, a compression impulse, and a restitution impulse. The compression impulse P_c is measured from the time the two objects collide to the time their relative velocity is zero. The restitution impulse P_r is measured from the time the relative velocity is zero to the time that the two objects separate. Therefore, $P_n = P_c + P_r$. The *Poisson Hypothesis* states that

$$e = \frac{P_r}{P_c} \text{ where } 0 \leq e \leq 1 \qquad (21)$$

The number e is the *Coefficient of Restitution* and depends on the material properties of the objects.

If the objects collide obliquely, there will generally be an initial phase during which the objects are sliding across one another. In this case

$$|P_b| = \mu |P_n| \qquad (22)$$

where μ is the coefficient of friction at the point of contact. Otherwise,

$$|P_b| < \mu |P_n| \qquad (23)$$

and is equal to the amount of the impulse necessary to prevent sliding.

Denote the velocities of the puck and mallet as

$$V_{puck} = (\dot{b}_p, \dot{n}_p, \dot{\theta}_p)^T \qquad (24)$$

$$V_{mallet} = (\dot{b}_m, \dot{n}_m, 0)^T \qquad (25)$$

The puck velocities are governed by the impulse momentum equations (1)-(3) while the mallet velocities are constant throughout the impact event.

The velocites of the objects at the contact point are given by

$$\dot{b}_{pc} = \dot{b}_p + r\dot{\theta}_p \qquad (26)$$

$$\dot{n}_{pc} = \dot{n}_p \qquad (27)$$

$$\dot{b}_{mc} = \dot{b}_m \qquad (28)$$

$$\dot{n}_{mc} = \dot{n}_m \qquad (29)$$

We define the relative sliding velocity S and the relative compression velocity C at the point of contact as

$$S = \dot{b}_{pc} - \dot{b}_{mc} \qquad (30)$$

$$C = \dot{n}_{pc} - \dot{n}_{mc} \qquad (31)$$

Substituting Equations (1)-(3) into (30) and (31) we have

$$S = \dot{b}_{p0} + \dot{\theta}_{p0} + \tfrac{3}{m}P_b - \dot{b}_m \qquad (32)$$

$$C = \dot{n}_{p0} + \tfrac{1}{m}P_n - \dot{n}_m \qquad (33)$$

The Routh method determines the impulse components, P_b and P_n, from equations (32)-(33) graphically as follows. Setting $S = 0$ and $C = 0$ in (32) and (33) define the *Line of No Sliding* and the *Line of Maximum Compression*, respectively in the (P_b, P_n) or *Impulse Plane*. In the present case these lines are given by

$$P_b = -\tfrac{m}{3}(\dot{b}_{p0} + \dot{\theta}_{p0} - \dot{b}_m) \qquad (34)$$

$$P_n = -m(\dot{n}_{p0} - \dot{n}_m) \qquad (35)$$

We also define the so-called *Line of Termination*, T,

$$P_n = -(1+e)m(\dot{n}_{p0} - \dot{n}_m) \qquad (36)$$

and the *Line of Limiting Friction*

$$P_b = \mu P_n \qquad (37)$$

At the onset of the impact, the point P representing the impulse, is located at the origin in the impulse space and lies on the line of limiting friction given by (37). Movement along this line slows the relative velocites of the objects at the contact point. If P reaches the line of no sliding before the line of termination then P switches to the line of no sliding. The impact ends when P_n reaches the line of termination. Figure 3 shows three different lines of limiting

[1]This assumption is without loss of generality since allowing mallet rotation does not change the main conclusions of the paper

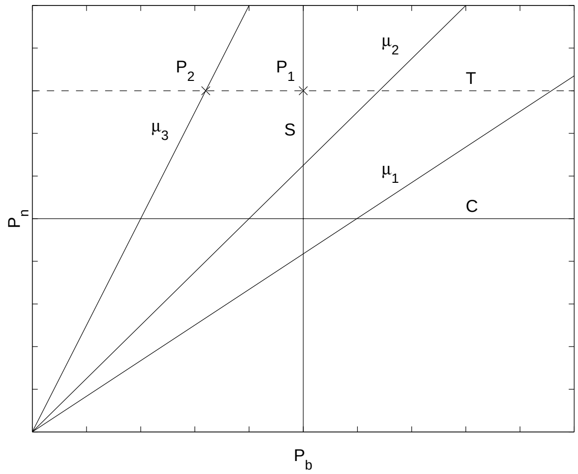

Figure 3: (P_b, P_n) plane for puck/wall impact

friction corresponding to three different values, $\mu_1 > \mu_2 > \mu_3$, for the coefficient of friction, μ. For $\mu = \mu_1$, the line of limiting friction intersects the line of no sliding before maximum compression. From this intersection point, we follow the line of no sliding and find that the impact terminates at the point P_1 shown. For lower values of friction, it may happen that the line of maximum compression is reached before the line of no sliding. The case $\mu = \mu_2$ shows this. In this case, the impact also terminates at point P_1. For still lower values of μ, it may happen that the line of limiting friction intersects the line of termination before the line of no sliding. This means that the friction is insufficient to stop sliding during the impact. The case $\mu = \mu_3$ shows this. In this case, the impact terminates at the point P_2.

Therefore, we have two possible solutions for the impulse imparted to the puck depending on whether or not sliding occurs throughout the impact. In the case that sliding terminates, i.e., in the case that the impact terminates at the point P_1, we have

$$P_b = -\frac{m}{3}(\dot{b}_{p0} + \dot{\theta}_{p0} - \dot{b}_m) \qquad (38)$$

$$P_n = -(1+e)m(\dot{n}_{p0} - \dot{n}_m) \qquad (39)$$

In the case that the friction is insufficient to prevent termination of sliding, so that the impact terminates at the point P_2, we have

$$P_b = -\mu(1+e)m(\dot{n}_{p0} - \dot{n}_m) \qquad (40)$$

$$P_n = -(1+e)m(\dot{n}_{p0} - \dot{n}_m) \qquad (41)$$

Equations (38) and (39) can be solved for the mallet velocities to achieve any desired impulse vector (P_b, P_n). Thus the entire reachable set of velocities given by Theorem 1 can be achieved by suitable choice of mallet velocities. In the case that sliding does no terminate we see that there is only one independent mallet velocity component in (40)-(41) and so only one component of the impulse can be arbitrarily specified. This introduces a further constraint on the set of achievable velocities. Combining Equations (19)-(20) with (40)-(41) this additional constraint can be expressed as

$$\frac{mr}{2}V_3 = \mu m(V_2 \cos\phi - V_1 \sin\phi) \qquad (42)$$

From Figure 3 we can see that the condition for termination of sliding depends on the relative positions of the line of limiting friction, line of no sliding, and line of maximum compression which, in turn depends on the initial contact velocities, and the coefficients of friction and restitution. In order for sliding to terminate before the impact terminates, we see from Figure 3 that the value of P_b given by (40) must be less than the value of P_b given by (38). This leads to the condition

$$|\dot{b}_{p0} + r_p\dot{\theta}_{p0} - \dot{b}_m| < 3\mu(1+e)|\dot{n}_{p0} - \dot{n}_m| \qquad (43)$$

For given values of the coefficients of friction and restitution, this no sliding condition relates the tangential velocity to the normal velocity at the contact point in the instant of impact.

5 Conclusions

We have produced a simple description of the achievable velocities of an air hockey puck subject to impacts with a mallet. We have seen that, if relative sliding between the mallet and puck terminates during the impact event, then Impulse Controllability and Impact Controllability are equivalent and the achievable set of puck velocities is a half space in $\Re^3$ whereas, if sliding does not terminate, an additional constraint must be satisfied which reduces the dimension of the reachable set by one. In this case, Impact Controllability is weaker than Impulse Controllability. We have an explicit condition on the relative velocities between the puck and mallet before impact that determines whether or not sliding will persist or terminate during the impact. These equations can be used both to predict the trajectories of the puck and to plan control strategies for a robotic air hockey system.

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